Mathematics Written Exam

- **Date:** May 27, 2019
- **Duration:** 2 hours
- **Instructions:**
  - This exam has two groups: I and II.
  - Group I has seven multiple choice questions and four possible answers to each one, of which only one is correct. In each question, circle the answer you think is correct, not showing the calculations made. If you circle more than one answer for the same question, the question will be considered unanswered. Each correct answer is worth 1 point, each incorrect answer is worth $-\frac{1}{3}$ points and each blank or invalid answer is worth 0 points. The minimum total number of points of this group is 0.
  - Group II has three open – ended questions, the first with four parts, the second with five parts and the third with two parts. The grade of each question is written before its text. Show all the calculations and justify all the reasonings made. If you need to round up numbers in intermediate steps, use two decimal places. Answer each question in the correspondent space and use the front and back of each sheet.
  - No questions will be answered during the exam. If you need to assume something while answering a question, state it, and be consistent with what you assumed in the steps that follow.
  - Only use writing material and a calculator. Do not use cell phones or other material.
  - Do not unstaple this exam.
  - The last two pages of this exam have formulae and space for drafts, whose contents will not be graded.

Name:

Group I

1 Let X be a random variable which may assume the values 0, 1 and 2. It is known that the average of X is 1 and that $P(X = 2 \mid X > 0) = \frac{2}{3}$. What is the value of $P(X = 0)$?

a) $\frac{1}{5}$

b) $\frac{2}{5}$

c) $\frac{2}{3}$

d) 0

2 Alberto, Bianca, Carlos and Diana go into a restaurant and are given a circular table with 4 chairs of different colors. As they cannot decide where to sit, they are going to draw the seats. What is the probability that Alberto and Bianca sit next to each other?

a) $\frac{1}{6}$

b) $\frac{1}{3}$

c) $\frac{1}{2}$

d) $\frac{2}{3}$

3 Let f be a real function of a real variable, with domain $\mathbb{R}$ and range $[-1,3]$. Which of the following statements is true?

a) The range of $|f|$ is $[0,1]$.

b) The graph of f may have vertical asymptotes.

c) The graph of f may have horizontal asymptotes.

d) f has no zeros.

4 Let f be a real function of a real variable, with domain $[1,5]$, continuous, strictly increasing and such that $f(1) = -6$ and $f(5) = 7$. Which of the following may be the domain of $\frac{1}{f}$?

a) $[1,5]$

b) $]1,5[$

c) $[1,3[\cup]3,5]$

d) $[1,2[\cup]2,4[\cup]4,5]$

5 Let f be a real function of a real variable, with domain $\mathbb{R}$, continuous, odd, with a single zero and such that $f(0) = 0$ and $f(3) > 0$. What is the solution set of the inequality below?

$$\frac{f(x)}{-1+e^{x^2-4}} \geq 0$$

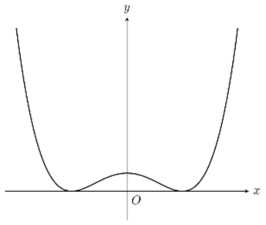
a) $]-2,0] \cup]2,+\infty[$

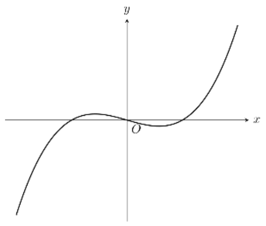
b) $]-\infty,-2] \cup [2,+\infty[$

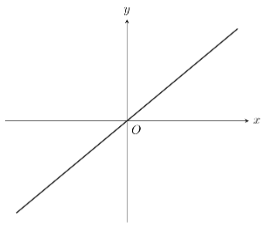
c) $[0,2]$

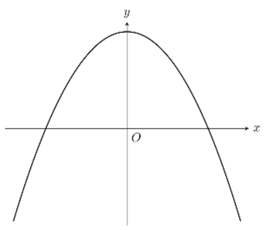
d) $[-2,0] \cup [2,+\infty[$

6 Let f be a real function of a real variable, with domain $\mathbb{R}$ and exactly 1 relative minimum and 1 relative maximum. Which of the following may be the graph of f', the derivative function of f?

a) 

b) 

c) 

d) 

7 Let z be a complex number such that the product of z by its conjugate is 2. What is the modulus of z?

a) 4

b) 2

c) $\sqrt{2}$

d) 1

Group II

1 **(5)** In a class with 10 students, there are 2 brothers. As a way of practicing debate skills, this class will organize a debate on the best answer to the question “should university tuitions be abolished?”. 3 students will be drawn to defend Yes and 3 students to defend No.

a) **(1)** What is the probability that the two brothers are drawn to defend the same answer? Show the result as a simplified fraction.

b) **(1,5)** Knowing the draw did not determine that the brothers should defend the same answer, what is the probability that both were left outside of the debate? Show the result rounded to two decimal places.

c) **(1)** After the debate was over, it was not possible to determine who was the winner, so it was decided to draw one. The draw consists on tossing a balanced coined several times until 2 heads in a row are obtained (and, in this case, Yes wins), or 2 tails in a row are obtained (and, in this case, No wins). What is the probability that the draw is decided with, at most, 3 coin tosses? Show the result as a simplified fraction.

d) **(1,5)** Consider the draw defined in part **c)** and events A: “The draw is decided with, at most, 3 coin tosses” and B: “In the first coin toss, heads is obtained”. State, justifying, if A and B are independent.

Answer Question 1

2 **(6)** Let f be a real function of a real variable, with domain $\mathbb{R}$, continuous, with zeros only in $x = -3$ and $x = 3$, and such that $\lim_{x \to 2^-} \left(f(x)\right) > 0$, $\lim_{x \to 2^-} \left(f(x) - \frac{1}{f(x)}\right) = 0$ and the derivative function of f, with domain $\mathbb{R}$, is defined by $f'(x) = -\frac{2}{5}x$. Let g be the real function of a real variable, with domain $\mathbb{R}$, defined by:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x < 2 \\ \dfrac{e^{x-2}}{x^2 - 3} & \text{if} \quad x \geq 2 \end{cases}$$

a) **(1)** Check if the graph of $\frac{1}{f}$ has vertical asymptotes.

b) **(1)** Find the zeros of g.

c) **(1,5)** Show that g is continuous in all its domain.

d) **(1,5)** Study the monotonicity and the existence of relative extrema of g.

e) **(1)** Let h be a real function of a real variable, with domain $\mathbb{R}$, continuous and such that $h(0) = 4$ and $h(1) = -5$. Show that $h \circ g$ has at least one zero in $[-3,2]$.

Answer Question 2

3 **(2)** In $\mathbb{C}$, the set of complex numbers, consider $w = -8 + 8\sqrt{3}i$.

a) **(1)** Find the area of the polygon whose vertices are the representation of the fourth roots of w in the complex plane.

b) **(1)** Let s be the fourth root of w whose representation is in the 4^{th} quadrant of the complex plane. Represent the zone of the complex plane defined by (remember that $|z|$ and $\arg(z)$ are, respectively, the modulus and the argument of z):

$$|s| \leq |z| \leq |w| \wedge \arg(w) < \arg(z) < \arg(s)$$

Answer Question 3

Formulae

Special Limits

$$\lim\left(\left(1+\frac{1}{n}\right)^n\right) = 1 \qquad (n \in \mathbb{N})$$

$$\lim_{a\to 0}\left(\frac{\sin(a)}{a}\right) = 1$$

$$\lim_{a\to 0}\left(\frac{e^a-1}{a}\right) = 1$$

$$\lim_{a\to 0}\left(\frac{\ln(a+1)}{a}\right) = 1$$

$$\lim_{a\to +\infty}\left(\frac{\ln(a)}{a}\right) = 0$$

$$\lim_{a\to +\infty}\left(\frac{b^a}{a^p}\right) = +\infty \qquad (b > 1, p \in \mathbb{R})$$

Derivation Rules

$$(a+b)' = a' + b'$$

$$(ab)' = a'b + ab'$$

$$\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}$$

$$(a^p)' = pa^{p-1}a' \qquad (p \in \mathbb{R})$$

$$(p^a)' = \ln(p)\, p^a a' \qquad (p \in \mathbb{R}^+ \setminus \{1\})$$

$$\left(\log_p(a)\right)' = \frac{a'}{\ln(p)\, a} \qquad (p \in \mathbb{R}^+ \setminus \{1\})$$

$$(\sin(a))' = a' \cos(a)$$

$$(\cos(a))' = -a' \sin(a) \qquad (b > 1, p \in \mathbb{R})$$

Complex Numbers

$$\left(\rho\operatorname{cis}(\theta)\right)^n = \rho^n \operatorname{cis}(n\theta) \qquad (n \in \mathbb{N})$$

$$\sqrt[n]{\rho\operatorname{cis}(\theta)} = \sqrt[n]{\rho}\operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad (n \in \mathbb{N}, k \in \{0, \ldots, n-1\})$$

Drafts